

1. (10%) Find all 3×3 diagonal matrices A satisfy $A^2 - 3A - 4I = 0$, where I is the 3×3 identity matrix.
2. (10%) Prove that in R^3 , the distance D between the point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

3. (10%) Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & 4 & 2 & -5 & -4 \end{bmatrix}$$

by row reduction.

4. (10%) Find the eigenvalues of A^{25} for

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

5. Let the vector space have the inner product

$$\langle p, q \rangle = \int_1^1 p(x)q(x)dx.$$

(a) (5%) Find $\|p\|$ for $p = x$.

(b) (5%) Find the distance between $p = 1$ and $q = x$.

6. (10%) If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m . Find all solutions to the system of congruences. That is, find all integers satisfying the following equations:

$$x \equiv 11 \pmod{15}$$

$$x \equiv 17 \pmod{21}$$

$$x \equiv 31 \pmod{35}$$

7. A bowl contains 15 red balls and 15 blue balls. A woman selects balls at random without looking at them.

(a) (5%) How many balls must she select to be sure of having at least 6 balls of the same color?

(b) (5%) How many balls must she select to be sure of having at least 6 blue balls?

8. (5%) A simple graph is called regular if every vertex of this graph has the same degree. A regular graph G of degree m has n vertices. The complementary graph $\overline{G}$ of G has the same vertices as G . Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G . For which values of m and/or n does G have the same number of edges as $\overline{G}$?

9. (16%) Determine the properties of the following binary relations on positive integers. Fill in the blank of the following table with O if the relation has the property, and otherwise with X.

$$R1 = \{(a,b) \mid a \geq b\};$$

$$R2 = \{(a,b) \mid a - b = 5k \text{ for some integer } k\};$$

$$R3 = \{(a,b) \mid a = kb \text{ for some integer } k\};$$

$$R4 = \{(a,b) \mid a = kb^2 \text{ for some integer } k\}.$$

	reflexive	symmetric	Anti-symmetric	Transitive
R1				
R2				
R3				
R4				

10. Solve the following recurrence equations.

(a) (5%) $T(0)=2$, $T(1)=5$, and $T(n)=5T(n-1)-6T(n-2)$ for $n>1$.

(b) (4%) $f(1)=1$ and $f(n)=f(n/2)+n$ for $n>1$.